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Commutators for Fourier multipliers on Besov Spaces[☆]

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Abstract

If T is any bounded linear operator on Besov spaces $B_p^{\sigma_j,q_j}(\mathbf{R}^n)(j=0,1,$ and $0<\sigma_1<\sigma<\sigma_0)$, it is proved that the commutator $[T,T_\mu]=TT_\mu-T_\mu T$ is bounded on $B_p^{\sigma,q}(\mathbf{R}^n)$, if T_μ is a Fourier multiplier such that μ is any (possibly unbounded) symbol with uniformly bounded variation on dyadic coronas.

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1. Introduction

The mapping properties of commutators $[T,M]=TM-MT$, for operators between function spaces, and their various generalizations play an important role in harmonic analysis, PDE, interpolation theory and other related areas.

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A typical situation arises when $M = M_b$ is the pointwise multiplication by a function b and T is a Calderón–Zygmund operator on $\mathbf{R}^n$. Then well-known results of Calderón [4] and Coifman et al. [8] state, respectively, that the commutator is bounded from L^2 into $W^{1,2}$ if b is a Lipschitz function, and from L^p into L^p if $1 < p < \infty$ and $b \in BMO$. Recently, its boundedness between some other function spaces, including Besov–Lipschitz and Triebel–Lizorkin spaces, has been extensively studied (see [13] and the references therein).

A related situation appears when $M = T_\mu$ is the Fourier multiplier with symbol μ, i.e. $\widehat{T_\mu f} = \mu \hat{f}$, where $\hat{f}$ is the Fourier transform of f. It was proved in [7] that, for Besov spaces of periodic functions $B_p^{\sigma,q}(\mathbf{T})$, the commutator

$$[T, T_\mu] : B_p^{\sigma,q}(\mathbf{T}) \to B_p^{\sigma,q}(\mathbf{T})$$

is bounded for a wide class of operators T and symbols μ (not necessarily bounded). The fact that the symbol μ is a sequence of complex numbers was used in the proof.

In this paper, we will deal with Besov spaces on $\mathbf{R}^n$. Now μ needs not be a sequence of complex numbers and Theorem 3 of Cerdà et al. [7] cannot be applied. However, the simple device of taking averages (see Lemma 2 below) allows to obtain a commutator theorem (Theorem 1) for $[T, T_\mu]$ for Besov spaces $B_p^{\sigma,q} = B_p^{\sigma,q}(\mathbf{R}^n)$ where, as in the periodic case, the boundedness assumption on μ is not required.

The description of Besov classes as approximation spaces, the calculation of almost optimal approximation elements in combination with real interpolation and the cancellation properties of the commutators will be the main tools used in the proof. See [5,6,9] and the references therein for commutator theorems related with the main interpolation methods.

We briefly summarize the contents of this paper. In Section 2, we include the needed definitions and background. Section 3 deals with admissible multipliers defined via an appropriate notion of variation on dyadic coronas, and Section 4 contains Theorem 1, the main result of this paper.

If A and B are two Banach spaces, we write $T : A \to B$ to mean that T is a bounded linear operator between A and B.

Finally, $P \preccurlyeq Q$ means that $P \leqslant cQ$ for some constant $c > 0$ independent of the variables involved, while by $P \simeq Q$ we mean that $P \preccurlyeq Q$ and $Q \preccurlyeq P$.

2. Preliminaries

Let us now start by briefly recalling some results about real interpolation theory and Besov spaces (see [1–3] for more details and definitions concerning interpolation theory and [1,2,10,11] for general properties of Besov spaces).

If $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$, for a given Banach couple $\bar{A} = (A_0, A_1)$, the corresponding interpolation Banach space is

$$\bar{A}_{\theta,q} = \{x \in \Sigma(\bar{A}) = A_0 + A_1; ||x||_{\bar{A}_{\theta,q}} < \infty\}$$

with

$$||x||_{\bar{A}_{\theta,q}} := ||t^{-\theta}K(t,x)||_{L^q(dt/t)},$$

where $K(t,x) = K(t,x;\bar{A}) := \inf\{||x_0||_{A_0} + t||x_1||_{A_1}; x = x_0 + x_1\}$ is the Peetre's K-functional.

If $\bar{A}$, $\bar{B}$ are two Banach couples, we denote by $\mathscr{L}(\bar{A};\bar{B})$ the set all linear operators $T:\Sigma(\bar{A})\to\Sigma(\bar{B})$ such that $T(A_j)\subset B_j$ $(j=0,1)$ and $||T|| = \max(||T||_{A_0,B_0};||T||_{A_1,B_1})<\infty$. If $T\in\mathscr{L}(\bar{A};\bar{B})$, then $T:\bar{A}_{\theta,q}\to\bar{B}_{\theta,q}$.

If

$$Sf(t) := \int_0^t f(s)\frac{ds}{s} + t\int_t^\infty f(s)\frac{ds}{s^2}$$

is the Calderón operator, we set

$$\sigma(\bar{A}) := \{x\in\Sigma(\bar{A}); ||x||_{\sigma(\bar{A})} := S(K(\cdot,x))(1)<\infty\}.$$

Observe that $\sigma(\bar{A})$ is a linear subspace of $\Sigma(\bar{A})$ which contain all real interpolation spaces $\bar{A}_{\theta,q}$ and moreover $||Tx||_{\sigma(\bar{B})}\leqslant||T||\,||x||_{\sigma(\bar{A})}, (T\in\mathscr{L}(\bar{A};\bar{B}))$.

Given $r>0$, let $V(r) = \{g\in\mathscr{S}' : \operatorname{supp}\hat{g}\subset[-r,r]^n\}$ $(V(0) = \{0\})$, where $\hat{g}$ is the Fourier transform of the distribution g.

The Besov space $B_p^{\sigma,q}$ (or $B_p^{\sigma,q}(\mathbf{R}^n)$), with $0<\sigma<\infty$ and $1\leqslant q,p<\infty$, is defined by

$$B_p^{\sigma,q} = \left\{f\in L^p(\mathbf{R}^n); ||f||_{B_p^{\sigma,q}} = \left(\int_0^\infty [r^\sigma E(r,f)]^q\frac{dr}{r}\right)^{1/q}<\infty\right\},$$

with $E(r,f) := \inf_{g\in V(r)}||f-g||_{L^p}$.

Real interpolation for couples of Besov spaces is described in the following Lemma (cf. [2, Theorem 6.4.5]).

Lemma 1. *Let* $0<\sigma_0,\ \tilde{\sigma}_0<\infty,\ 1\leqslant p,q,q_0,q_q,r<\infty,\ 0<\theta<1$ *and* $\sigma = (1-\theta)\sigma_0 + \theta\tilde{\sigma}_0$. *Then*

$$(B_p^{\sigma_0,q_0}, B_p^{\tilde{\sigma}_0,q_1})_{\theta,q} = B_p^{\sigma,q}, \quad (L^p, B_p^{\sigma_0,r})_{\theta,q} = B_p^{\theta\sigma_0,q}.$$

The calculation of almost optimal approximation elements is contained in the following proposition.

Proposition 1. *Let* $0<\sigma_0,\tilde{\sigma}_0<\infty,\ 1\leqslant p,q_0,\tilde{q}_0<\infty$, *and assume that* $\rho := \sigma_0 - \tilde{\sigma}_0>0$. *Then*

$$K(t^\rho,f;B_p^{\sigma_0,q_0},B_p^{\tilde{\sigma}_0,\tilde{q}_0})\simeq||P_tf||_{B_p^{\sigma_0,q_0}} + t^\rho||f-P_tf||_{B_p^{\tilde{\sigma}_0,\tilde{q}_0}}$$

and

$$K(t^\sigma,f;B_p^{\sigma,q},L^p)\simeq||P_tf||_{B_p^{\sigma,q}} + t^\sigma||f-P_tf||_{L^p},$$

where P_t *is the Fourier multiplier with symbol* $\chi_{[-t,t]^n}$.

Proof. Let $g_t \in V(t)$ be such that $||f - g_t||_{L^p} \leqslant 2E(t,f)$. Since (see [12, IV, Theorem 4]) $||P_t f||_{L^p} \leqslant c_p ||f||_{L^p}$ $(\forall f \in L^p(\mathbf{R}^n),\ t>0)$ we have that

$$||f - P_t f||_{L^p} \leqslant ||f - g_t||_{L^p} + ||g_t - P_t f||_{L^p} = ||f - g_t||_{L^p} + ||P_t(g_t - f)||_{L^p} \leqslant CE(t,f).$$

Hence Theorem 4 of [7] applies and

$$K(t^\rho, f; B_p^{\sigma_0,q_0}, B_p^{\tilde{\sigma}_0,\tilde{q}_0}) \simeq ||P_t f||_{B_p^{\sigma_0,q_0}} + t^\rho ||f - P_t f||_{B_p^{\tilde{\sigma}_0,\tilde{q}_0}}. \qquad \square$$

3. Admissible multipliers

Let $R = \prod_{k=1}^n [a_k, b_k]$ be a rectangle on $\mathbf{R}^n$ with sides parallel to the axes and let m be a function defined on R. We define Δ_R by

$$\Delta_R(m) = \Delta_{h_1}^{(1)} \Delta_{h_2}^{(2)} \cdots \Delta_{h_n}^{(n)} m(a_1, \dots, a_n),$$

where $h_k = b_k - a_k$ and $\Delta^{(k)}$ is the difference operator in the kth variable, i.e.

$$\Delta_h^{(k)} m(a_1, \dots, a_n) = m(a_1, \dots, a_{k-1}, a_k + h_k, a_{k+1}, \dots, a_n) - m(a_1, \dots, a_n).$$

Given $j \geqslant 0$, let $Q_j = [-2^j, 2^j]^n$. We denote by $C_j = Q_j \backslash Q_{j-1}$ $(C_0 = Q_1)$ the j-dyadic corona, and by $\bar{C}_j$ the closed j-dyadic corona.

Following [14], the space of functions of bounded variation on $\bar{C}_j$ is defined inductively in n as follows:

If $n = 1$, we say that m is of bounded variation on $\bar{C}_j$ if

$$\sup_\pi \sum |m(t_k) - m(t_{k-1})| < \infty,$$

with the sup taken over all partitions π of $\bar{C}_j = [-2^j, -2^{j-1}] \cup [2^{j-1}, 2^j]$, $(\bar{C}_0 = [-1, 1])$.

For $n \geqslant 2$, we say that m is of bounded variation on $\bar{C}_j$ if the following properties are satisfied:

(i) We have that

$$\sup_{\mathscr{R}} \sum_{R \in \mathscr{R}} |\Delta_R(m)| < \infty,$$

where the sup runs over all families R of rectangles with sides parallel to the axes of disjoint interior whose vertices belong to $\bar{C}_j$.

(ii) For each $1 \leqslant k \leqslant n-1$ the function $m(x_1, \dots, x_k, 2^j, \dots, 2^j)$ considered as a function of the first k variables, is of bounded variation on the k-dimensional rectangle $R = \prod_{j=1}^k [-2^j, 2^j]$.

(iii) The condition analogous to (ii) is valid for every one of the $n!$ permutations of the variables $x_1, \dots, x_n$.

We denote by $||m||_{V(\bar{C}_j)}$ the sum of all quantities appearing in (i)–(iii).

Definition 1. An *admissible multiplier* will be a function $\mu: \mathbf{R}^n \to \mathbf{C}$ such that

$$V(\mu) := \sup_{j \geqslant 0} ||\mu||_{V(\bar{C}_j)} < \infty. \tag{1}$$

Notice that $\log^+(\max\{|x_1|, \dots, |x_n|\})$ is a simple example of unbounded admissible multiplier, but $\mu(x_1, \dots, x_n) = \log^+(\max\{|x_1|, \dots, |x_n|\})$ if $x_j > 0$ $(j = 1, \dots, n)$ and $\mu(x_1, \dots, x_n) = 0$ otherwise, is not admissible.

Definition 2. A *dyadic multiplier* will be a function

$$\mu = \{\mu_j\}_{j \geqslant 0} := \sum_{j=0}^{\infty} \mu_j \chi_{C_j},$$

which is constant on every corona C_j.

For dyadic multipliers the admissibility condition (1) takes the form

$$V(\mu) = \sup_{j \geqslant 0} |\mu_j - \mu_{j-1}| < \infty \quad (\mu_{-1} = 0)$$

and for any admissible multiplier, μ, we claim that

$$\mu^{(d)} := \sum_{j=0}^{\infty} \mu(2^j, \dots, 2^j)\chi_{C_j} = \sum_{j=0}^{\infty} \mu_j \chi_{C_j}$$

defines an admissible dyadic multiplier with

$$V(\mu^{(d)}) \leqslant V(\mu). \tag{2}$$

Since the case $n = 2$ is already completely typical situation of the general case, to avoid some notational complications, let us to prove this claim only in the case.

If $j \geqslant 1$, considering the rectangle $R = [2^{j-1}, 2^j] \times [2^{j-1}, 2^j]$ we have that

$$\begin{aligned}
|\mu_j - \mu_{j-1}| &\leqslant |\mu(2^{j-1}, 2^{j-1}) - \mu(2^j, 2^{j-1}) + \mu(2^j, 2^j) - \mu(2^{j-1}, 2^j)| \\
&\quad + |\mu(2^j, 2^{j-1}) - \mu(2^j, 2^j)| + |\mu(2^{j-1}, 2^j) - \mu(2^j, 2^j)| \\
&= |\Delta^{(1)}_{2^{j-1}} \Delta^{(2)}_{2^{j-1}} \mu(2^{j-1}, 2^{j-1})| + |\Delta^{(2)}_{2^{j-1}} \mu(2^j, 2^{j-1})| \\
&\quad + |\Delta^{(1)}_{2^{j-1}} \mu(2^{j-1}, 2^j)| \\
&\leqslant ||\mu||_{V(\bar{C}_j)}.
\end{aligned}$$

Similarly, if $j = 0$, considering $R = [-1, 1] \times [-1, 1]$, we get

$$|\mu_0| \leqslant |\mu_0 - \mu(0,0)| + |\mu(0,0)| \leqslant ||\mu||_{V(\bar{C}_0)} + |\mu(0,0)|.$$

Finally, we recall that a *dyadic interval* of $\mathbf{R}$ will be one from the sequences $\{[2^k, 2^{k+1}]\}_{k \in \mathbb{Z}}, \{[-2^{k+1}, -2^k]\}_{k \in \mathbb{Z}}$, and that by a *dyadic rectangle* on $\mathbf{R}^n$ we mean a rectangle R which is the product of n dyadic intervals. We shall denote by $\mathscr{D}$ the family of the dyadic rectangles.

4. The main result

Theorem 1. *Let* $1 \leqslant q, q_0, q_1, \tilde{q}_0, \tilde{q}_1 < \infty, \sigma_0 > \tilde{\sigma}_0 > 0, \sigma_1 > \tilde{\sigma}_1 > 0$, *such that* $\sigma_0 - \tilde{\sigma}_0 = \sigma_1 - \tilde{\sigma}_1$, $\sigma = (1-\theta)\sigma_0 + \theta\tilde{\sigma}_0$, $\tilde{\sigma} = (1-\theta)\sigma_1 + \theta\tilde{\sigma}_1$ $(0<\theta<1)$, *and assume that* $1<p<\infty$.

If μ *is an admissible multiplier, then*

$$||[T, T_\mu]||_{\mathscr{L}(B_p^{\sigma,q};B_p^{\tilde{\sigma},q})} \leqslant c||T||,$$

where T *is any bounded linear operator between the couples of Besov spaces* $(B_p^{\sigma_0,q_0}; B_p^{\tilde{\sigma}_0,\tilde{q}_0})$ *and* $(B_p^{\sigma_1,q_1}; B_p^{\tilde{\sigma}_1,\tilde{q}_1})$, *and*

$$||T|| = \max(||T||_{\mathscr{L}(B_p^{\sigma_0,q_0};B_p^{\sigma_1,q_1})}, ||T||_{\mathscr{L}(B_p^{\tilde{\sigma}_0,\tilde{q}_0};B_p^{\tilde{\sigma}_1,\tilde{q}_1})}).$$

In order to prove the main theorem, let us start with a reduction to dyadic multipliers.

Lemma 2. *Let* μ *be an admissible multiplier and let* $\mu^{(d)}$ *its admissible dyadic multiplier, i.e.*

$$\mu^{(d)} := \sum_{j=0}^{\infty} \mu(2^j, \dots, 2^j)\chi_{C_j} = \sum_{j=0}^{\infty} \mu_j \chi_{C_j}.$$

If T *is any bounded linear operator from* $B_p^{\sigma,q}$ *to* $B_p^{\tilde{\sigma},q}$, *then*

$$[T, T_{\mu(d)}] : B_p^{\sigma,q} \to B_p^{\tilde{\sigma},q}$$

if and only if

$$[T, T_\mu] : B_p^{\sigma,q} \to B_p^{\tilde{\sigma},q}.$$

Proof. Again in order to avoid some notational complications, let us assume that $n = 2$.

First, we are going to see that $\tilde{\mu} = \mu - \mu^{(d)}$ is bounded. Given $(x_1, x_2) \in \mathbf{R}^2$, let $j \geqslant 0$ be such that $(x_1, x_2) \in C_j$. Then, if $x_1 = 2^j$,

$$|\tilde{\mu}(2^j, x_2)| = |\mu(2^j, x_2) - \mu(2^j, 2^j)| \leqslant ||\mu||_{V(\bar{C}_j)} \leqslant V(\mu).$$

Similarly, if $x_2 = 2^j$, $|\tilde{\mu}(x_1, 2^j)| \leqslant V(\mu)$. Finally, if (x_1, x_2) is an interior point of C_j, by considering the rectangle R of vertices (x_1, x_2), $(2^j, x_2)$, $(2^j, 2^j)$, $(x_1, 2^j)$,

$$\begin{aligned}|\tilde{\mu}(x_1,x_2)| &\leqslant |\mu(x_1,x_2) - \mu(2^j,x_2) + \mu(2^j,2^j) - \mu(x_1,2^j)| \\ &\quad + |\mu(2^j,x_2) - \mu(2^j,2^j)| + |\mu(x_1,2^j) - \mu(2^j,2^j)| \\ &\leqslant ||\mu||_{V(\bar{C}_j)} \leqslant V(\mu).\end{aligned}$$

On the other hand, for any dyadic rectangle R, there is $j \geqslant 0$ such that $R \subset \bar{C}_j$ and then, by (2),

$$||\tilde{\mu}||_{V(R)} \leqslant ||\mu||_{V(R)} + ||\mu^{(d)}||_{V(R)} \leqslant ||\mu||_{V(\bar{C}_j)} + ||\mu^{(d)}||_{V(\bar{C}_j)} \leqslant 2V(\mu).$$

Thus

$$\sup_{R \in \mathscr{D}} ||\tilde{\mu}||_{V(R)} \leqslant 2V(\mu)$$

and $T_{\tilde{\mu}} : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$, by the Marcinkiewicz Multiplier Theorem (see [14]). Hence

$$T_{\tilde{\mu}} : B_p^{\sigma,q} \to B_p^{\sigma,q},$$

since $E(t, T_{\tilde{\mu}}f) \leqslant \inf_{g \in V(t)} ||T_{\tilde{\mu}}f - T_{\tilde{\mu}}g||_{L^p} \leqslant ||T_{\tilde{\mu}}||E(t,f)$, and the proof ends by observing that $[T, T_\mu] = [T, T_{\tilde{\mu}}] + [T, T_{\mu^{(d)}}]$. □

Proof of Theorem 1. By the previous lemma, we may assume that μ is a dyadic admissible multiplier, i.e. $\mu = \sum_{k=0}^{\infty} \mu_k \chi_{C_k}$, with $\mu_0 = 0$ and $\sup_k |\mu_{k+1} - \mu_k| < \infty$. Then, formally,

$$T_\mu f = \sum_{k=1}^{\infty} \mu_k (P_{2^k} f - P_{2^{k-1}} f), \tag{3}$$

since $\mu_k (P_{2^k} f - P_{2^{k-1}} f)^\wedge = \mu_k (\chi_{Q_k} - \chi_{Q_{k-1}}) \hat{f} = \mu_k \chi_{C_k} \hat{f}$.

Now, by denoting

$$\lambda_0 = \mu_1 - \mu_0 = \mu_1, \quad \lambda_1 = \mu_2 - \mu_1, \ldots, \lambda_k = \mu_{k+1} - \mu_k, \ldots$$

we get $\sup_k |\lambda_k| = \sup_k |\mu_{k+1} - \mu_k| < \infty$, since $\lambda_0 = \mu_1$, $\lambda_0 + \lambda_1 = \mu_2, \ldots, \sum_{j=0}^{k} \lambda_j = \mu_{k+1}, \ldots$

Then

$$T_\mu f = \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k-1} \lambda_j \right) (P_{2^k} f - P_{2^{k-1}} f) = \sum_{j=0}^{\infty} \lambda_j \sum_{k>j} (P_{2^k} f - P_{2^{k-1}} f)$$

yields $T_\mu f = \sum_{j=0}^{\infty} \lambda_j (f - P_{2^j} f)$, a convergent series. Moreover

$$T_\mu : \sigma(\bar{A}) \to \Sigma(\bar{A}), \quad \text{if } \bar{A} = (B_p^{\sigma_0,q_0}; B_p^{\tilde{\sigma}_0,\tilde{q}_0}), \tag{4}$$

since by Proposition 1, if $\rho = \sigma_0 - \tilde{\sigma}_0$, we have that

$$\|T_\mu f\|_{\Sigma(\bar{A})} \leqslant \|\lambda\|_\infty \sum_{j\geqslant 0} \|f - P_{2^j} f\|_{B_p^{\tilde{\sigma}_0,\tilde{q}_0}} \preccurlyeq \|\lambda\|_\infty \sum_{j\geqslant 0} \frac{K(2^{\rho j}, f; \bar{A})}{2^{\rho j}}$$
$$\leqslant \frac{2^\rho \|\lambda\|_\infty}{\rho \log 2} \int_1^\infty \frac{K(s, f; \bar{A})}{s} \frac{ds}{s} \leqslant \frac{2^\rho \|\lambda\|_\infty}{\rho \log 2} \|f\|_{\sigma(\bar{A})}.$$

Similarly, $T_\mu : \sigma(\bar{B}) \to \Sigma(\bar{B})$, $\bar{B} = (B_p^{\sigma_1,q_1}; B_p^{\tilde{\sigma}_1,\tilde{q}_1})$.

Now, given $T \in \mathscr{L}(\bar{A}; \bar{B})$ and $f \in \sigma(\bar{A})$, it follows from (4) that

$$TT_\mu f = \sum_{j=0}^{\infty} \lambda_j (Tf - TP_{2^j} f) \quad (\text{convergence in } \Sigma(\bar{B})).$$

Moreover $Tf \in \sigma(\bar{B})$ since, by interpolation, $T : \sigma(\bar{A}) \to \sigma(\bar{B})$. Then $T_\mu T_f = \sum_{j=0}^{\infty} \lambda_j (Tf - P_{2^j} Tf)$ in $\Sigma(\bar{B})$. Hence, $[T, T_\mu] : \sigma(\bar{A}) \to \Sigma(\bar{B})$ and

$$[T, T_\mu] f = \sum_{j=0}^{\infty} \lambda_j (Tf - TP_{2^j} f) - \sum_{j=0}^{\infty} \lambda_j (Tf - P_{2^j} f).$$

Obviously $(Tf - TP_{2^j} f) - (Tf - P_{2^j} Tf) = P_{2^j} Tf - TP_{2^j} f$ (here is where cancellation takes place) and we may decompose $[T, T_\mu] f$ in two sums,

$$[T, T_\mu] f = \sum_{\Theta} \lambda_j (P_{2^j} Tf - TP_{2^j} f) + \sum_{\mathbf{N}\setminus\Theta} \lambda_j (Tf - TP_{2^j} f) - \lambda_j (Tf - P_{2^j} Tf),$$

where $\Theta = \Theta(t) \coloneqq \{j \geqslant 0; 2^{\rho(j+1)} < t\}$.

Applying Proposition 1 to Tf and f, since $\rho = \sigma_0 - \tilde{\sigma}_0 = \sigma_1 - \tilde{\sigma}_1$, we get

$$\|\lambda_j (P_{2^j} Tf - TP_{2^j} f)\|_{\sigma_1, q_1} \leqslant \|\lambda\|_\infty (\|P_{2^j} Tf\|_{B_p^{\sigma_1,q_1}} + \|TP_{2^j} f\|_{B_p}^{\sigma_1,q_1})$$
$$\preccurlyeq \|\lambda\|_\infty \|T\| K(2^{\rho j}, f; \bar{A}).$$

Similarly,

$$\|\lambda_j (Tf - TP_{2^j} f) - \lambda_j (Tf - P_{2^j} Tf)\|_{B_p^{\tilde{\sigma}_1,\tilde{q}_1}} \preccurlyeq \|\lambda\|_\infty \|T\| \frac{K(2^{\rho j}, f; \bar{A})}{2^{\rho j}}.$$

Thus

$$\begin{aligned}K(t,[T,T_\mu]f;\bar{B}) &\leqslant \left\|\sum_{\Theta}\lambda_j(P_{2^j}Tf-TP_{2^j}f)\right\|_{B_p^{\sigma_1,q_1}}\\ &\quad + t\left\|\sum_{N\setminus\Theta}\lambda_j(Tf-TP_{2^j}f)-\lambda_j(Tf-P_{2^j}Tf)\right\|_{B_p^{\tilde{\sigma}_1,\tilde{q}_1}}\\ &\preccurlyeq \left(\sum_{\Theta}K(2^{\rho j},f;\bar{A})+t\sum_{N\setminus\Theta}\frac{K(2^{\rho j},f;\bar{A})}{2^{\rho j}}\right)\\ &\preccurlyeq \left(\int_0^t K(s,f;\bar{A})\frac{ds}{s}+t\int_t^\infty \frac{K(s,f;\bar{A})}{s}\frac{ds}{s}\right)\\ &= S(K(\cdot,f;\bar{A}))(t)\end{aligned}$$

and, by Minkowski inequality and Hardy's inequalities for averages (see [1]), $||[T,T_\mu]f||_{\bar{B}_{\theta,q}}\leqslant c||f||_{\bar{A}_{\theta,q}}$, with $\bar{A}_{\theta,q}=B_p^{\sigma,q}$ and $\bar{B}_{\theta,q}=B_p^{\tilde{\sigma},q}$ by Lemma 1, and $c=c(\rho,||\lambda||_\infty,||T||)$. $\square$

Remark 1. Once the reduction to dyadic multipliers has been established, using (3), Theorem 3 of [7] could be possibly adapted to end the proof of Theorem 1. We have preferred to include here an easier direct proof that does not use the abstract methods of [6] and that, in combination with Lemma 2, can be also used in the periodic to give a short proof of the commutator theorem presented in [7], including the case of several variables.

Remark 2. The admissibility conditions on the multipliers cannot be weakened if $[T,T_\mu]$ has to be bounded for the elementary operator $Tf(x)=f(2x)$, since if μ is dyadic and $[T,T_\mu]$ is bounded when $Tf(x)=f(2x)$, by considering f such that $||f||_{B_p^{\sigma,q}}=1$ and $\operatorname{supp}\hat{f}\subset C_k$, then

$$||[T,T_\mu]f||_{B_p^{\sigma,q}}\simeq|\mu_{k+1}-\mu_k|$$

and $\sup_k|\mu_{k+1}-\mu_k|<\infty$.

Remark 3. By Proposition 1, we may also consider the Banach couples (B_p^{σ,q_0},L^P) and (B_q^{σ,q_1},L^q), but reiteration would lead to the same result.

Remark 4. A commutator result similar to Theorem 1 holds for Besov spaces $B_X^{\sigma,q}$ when X is any r.i. space with Boyd indices strictly between 0 and 1, since what is really needed in Proposition 1 is the uniform boundedness of P_t on X, which is an interpolation space between two L^p spaces with $1<p<\infty$.

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